Analysis of mass difference of the π and ρ with Bethe-Salpeter equation

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Abstract

In this article, we take into account the one-pion exchange force besides the one-gluon exchange force to study the mass difference of the π and ρ mesons with the Bethe-Salpeter equation. After projecting the Bethe-Salpeter equation into an simple form, we can see explicitly that the bound energy $|E_\pi| \gg |E_\rho|$.

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1 Introduction

The constituent quark models have given many successful descriptions of the hadron spectroscopy, the simple constituent quark mass plus hyperfine spin-spin interaction model works well for the ground state mesons [1],

$$M_m = M_1 + M_2 + C\frac{\vec{\sigma}_1 \cdot \vec{\sigma}_2}{M_1 M_2}\,, \tag{1}$$

where the coefficient C can be fitted phenomenologically. The masses of the ground state pseudoscalar and vector mesons are $M_\pi = 140\,\mathrm{MeV}$, $M_K = 494\,\mathrm{MeV}$, $M_\eta = 548\,\mathrm{MeV}$, $M_{\eta'} = 958\,\mathrm{MeV}$, $M_\rho = 775\,\mathrm{MeV}$, $M_\omega = 783\,\mathrm{MeV}$, $M_{K^*} = 892\,\mathrm{MeV}$, and $M_\phi = 1019\,\mathrm{MeV}$ [2]. The mass difference between the π and ρ mesons are huge, we have to resort to large hyperfine spin-spin interactions for explanation. The fine spin-orbit interactions and the hyperfine spin-spin interactions are usually studied in the relativized quark model based on the one-gluon exchange plus linear confinement potential motivated by QCD [3], for more literatures, one can consult the comprehensive review Ref.[4]. One may wander why the contributions from the hyperfine interactions in Eq.(1) are so large, and how to understand them in the quantum field theory.

At the energy scale $\mu = 4\pi f_\pi \approx 1\,\mathrm{GeV}$, the approximate chiral $SU_L(3) \times SU_R(3)$ symmetry is spontaneously broken to the $SU_V(3)$ symmetry by the small current quark masses m_u, m_d and m_s, and there appear eight Nambu-Goldstone bosons (in the following we will neglect the word Nambu for simplicity). The masses of the Goldstone bosons are related with the current quark masses through the Gell-Mann-Oakes-Renner relation [5]. The quark fields $q(x)$ are usually decomposed as

$$q(x) = \exp\left[-i\gamma_5 \xi^a(x)\lambda^a\right] \widetilde{q}(x)\,, \tag{2}$$

where the $\widetilde{q}(x)$ and $\xi^a(x)$ denote the constituent quark fields (or Goldstone free fields) and the octet Goldstone boson fields, respectively [6], there exist interactions among the quarks and the Goldstone bosons. For example, the spectra of the nucleons, Δ resonances and

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the strange hyperons are well described by the constituent quark model with the harmonic confinement potential plus the one-Goldstone-boson exchanges induced potential [7]. In the energy region between the confinement and the spontaneous chiral symmetry breaking, the elementary degrees of freedom are quarks, gluons and Goldstone bosons [8].

On the other hand, the mass breaking effects in the chiral doublets are very large [2, 9], for example,

$$\begin{aligned} \pi(0^-, 140\,\mathrm{MeV}) &\leftrightarrow f_0(600)(0^+, 400-1200\,\mathrm{MeV})\,, \\ \rho(1^-, 775\,\mathrm{MeV}) &\leftrightarrow a_1(1260)(1^+, 1230\,\mathrm{MeV})\,, \\ p(\frac{1}{2}^+, 938\,\mathrm{MeV}) &\leftrightarrow N(1535)(\frac{1}{2}^-, 1525-1545\,\mathrm{MeV})\,, \end{aligned} \tag{3}$$

which requires that the chiral symmetry should be badly broken, here we use $\leftrightarrow$ to denote the chiral rotations. The chiral massive quark dresses itself with the gluon cloud and quark-antiquark pairs, and acquires a dynamically generated large mass. We usually carry out the re-summation of the loops nonperturbatively with the Dyson-Schwinger equation, and obtain the Euclidean constituent quark masses by the definition $p^2 = M^2(p^2)$, which are compatible with the values used in the constituent quark models [10, 11]. The light pseudoscalar mesons play a double role, as both Goldstone bosons and $q\bar{q}$ bound states.

The exchanges of the one-gluon and one-Goldstone-boson between the two constituent quarks result in the hyperfine interactions H_C and H_F, respectively [7, 12],

$$\begin{aligned} H_C &\sim \frac{1}{M_i M_j}\vec{\lambda}_i^C \cdot \vec{\lambda}_j^C \vec{\sigma}_i \cdot \vec{\sigma}_j\,, \\ H_F &\sim \frac{1}{M_i M_j}\vec{\lambda}_i^F \cdot \vec{\lambda}_j^F \vec{\sigma}_i \cdot \vec{\sigma}_j\,, \end{aligned} \tag{4}$$

and they both contribute to the spin-spin interactions. In this article, we take into account the contributions from the one-Goldstone-boson exchange force besides the one-gluon exchange force, study the π and ρ mass difference with the Bethe-Salpeter equation, and try to understand the difference in the quantum field theory.

The Bethe-Salpeter equation is a conventional approach in dealing with the two-body relativistic bound state problems [13], and has given many successful descriptions of the hadron properties in a Poincare covariant way [11, 14].

The article is arranged as follows: we solve the Bethe-Salpeter equation for the $u\bar{d}$ bound states in Sec.2; in Sec.3, we present the numerical results; and Sec.4 is reserved for our conclusions.

2 Bethe-Salpeter equation

We write down the ladder Bethe-Salpeter equation for the $u\bar{d}$ bound states in the Euclidean spacetime[2],

$$
\begin{aligned}
S_u^{-1}\left(q+\xi_u P\right)\chi(q,P)S_{\bar{d}}^{-1}\left(q-\xi_{\bar{d}}P\right) &= -\int \frac{d^4k}{(2\pi)^4}\left[\frac{\lambda^a}{2}\gamma_\mu\chi(k,P)\frac{\lambda^a}{2}\gamma_\mu\frac{g_s^2(q-k)}{(q-k)^2}\right. \\
&\quad \left. +i\gamma_5\chi(k,P)i\gamma_5\frac{g_\pi^2(q-k)}{(q-k)^2+m_\pi^2}\right], \qquad (5)\\
S_{u/\bar{d}}^{-1}\left(q\pm\xi_{u/\bar{d}}P\right) &= i\left(\gamma\cdot q\pm\xi_{u/\bar{d}}\gamma\cdot P\right)+M_{u/\bar{d}}\,, \\
\xi_{u/\bar{d}} &= \frac{M_{u/\bar{d}}}{M_u+M_{\bar{d}}}\,,
\end{aligned}
$$

the P_μ is the four-momentum of the center of mass of the $u\bar{d}$ bound states, the q_μ is the relative four-momentum between the u and $\bar{d}$ quarks, the $\chi(q,P)$ is the Bethe-Salpeter amplitude of the $u\bar{d}$ bound states, and the $g_\pi(q-k)$ and $g_s(q-k)$ are the energy dependent π-quark and gluon-quark coupling constants, respectively. In this article, we take the $g^2(k)$ as a modified Gaussian distribution, $g^2(k) = A\left(\frac{k^2}{\Lambda^2}\right)^2\exp\left(-\frac{k^2}{\Lambda^2}\right)$, where the strength A and the distribution width Λ^2 are free parameters. The ultraviolet behavior of the modified Gaussian distribution warrants that the integral in the Bethe-Salpeter equation is convergent.

The Euclidean Bethe-Salpeter amplitudes of the $u\bar{d}$ bound states can be decomposed as

$$
\begin{aligned}
\chi^\pi(q,P) &= \gamma_5\left\{F_\pi(q,P)+i\,\not{P}F_1^\pi(q,P)+i\,\not{q}q\cdot PF_2^\pi(q,P)+[\not{P},\not{q}]\,F_3^\pi(q,P)\right\}\,, \\
\chi^\rho(q,P) &= \not{\epsilon}\left\{iF_\rho(q,P)+\not{P}F_1^\rho(q,P)-\not{q}q\cdot PF_2^\rho(q,P)+i[\not{P},\not{q}]F_3^\rho(q,P)\right\} \\
&\quad +q\cdot\epsilon\left\{q\cdot PF_2^\rho(q,P)+2i\,\not{P}F_3^\rho(q,P)\right\} \\
&\quad +q\cdot\epsilon\left\{F_4^\rho(q,P)+i\,\not{P}q\cdot PF_5^\rho(q,P)-i\,\not{q}F_6^\rho(q,P)+[\not{P},\not{q}]F_7^\rho(q,P)\right\}\,, \qquad (6)
\end{aligned}
$$

due to Lorentz covariance [14, 15], where the ϵ_μ is the polarization vector of the ρ meson, the $F_\pi(q,P)$, $F_i^\pi(q,P)$, $F_\rho(q,P)$ and $F_i^\rho(q,P)$ are the components of the Bethe-Salpeter amplitudes, which can be expanded in terms of Tchebychev polynomials $T_n^{\frac{1}{2}}(\cos\theta)$ [16], where θ is the included angle between q_μ and P_μ. Numerical calculations indicate that taking only the terms $T_0^{\frac{1}{2}}(\cos\theta)=1$ can give satisfactory results [17]. If we take into account the small terms with $n\geq 1$, the predictions may be improved mildly. In the following, we use the amplitudes $F_{\pi/\rho}(q^2,P^2)$ and $F_i^{\pi/\rho}(q^2,P^2)$ to denote the $n=0$ terms of the Bethe-Salpeter amplitudes $F_{\pi/\rho}(q,P)$ and $F_i^{\pi/\rho}(q,P)$, respectively. Then the Bethe-Salpeter equations can be projected into four and eight coupled integral equations for the π and ρ mesons respectively, and it is very difficult to solve them numerically. Furthermore, we cannot obtain physical insight from those involved integral equations.

Multiplying both sides of the Bethe-Salpeter equations of the π and ρ mesons by $\gamma_5\,[\not{q},\not{P}]$ [18], and $\not{\epsilon}\,[\not{q},\not{P}]+[\not{q},\not{P}]\,\not{\epsilon}$, $q\cdot\epsilon q\cdot P\,\not{P}$ respectively, completing the trace in the

[2] In this article, we use the metric $\delta_{\mu\nu}=(1,1,1,1)$, $\{\gamma_\mu\gamma_\nu+\gamma_\nu\gamma_\mu\}=2\delta_{\mu\nu}$, the momentums $k_\mu=(k_4,\vec{k})$, $q_\mu=(q_4,\vec{q})$ and $P_\mu=(iE,\vec{P})$ with $P^2=-M^2_{\pi/\rho}$.

Dirac spinor space, carrying out the integrals for the included angle θ, and neglecting the small components F_3^π, F_3^ρ, F_5^ρ and F_6^ρ, we can obtain the following three relations,

$$\begin{aligned} F_\pi(q^2,P^2) - (M_u + M_{\bar d})\, F_1^\pi(q^2,P^2) &= 0\,, \\ F_\rho(q^2,P^2) + (M_u + M_{\bar d})\, F_1^\rho(q^2,P^2) &= 0\,, \\ 2F_\rho(q^2,P^2) - (M_u + M_{\bar d})\, F_4^\rho(q^2,P^2) &= 0\,. \end{aligned} \tag{7}$$

The Bethe-Salpeter amplitudes can be approximated as

$$\begin{aligned} \chi^\pi(q,P) &= \gamma_5 \left(1 + \frac{i \not P}{M_u + M_{\bar d}}\right) F_\pi(q^2,P^2)\,, \\ \chi^\rho(q,P) &= \left\{ \not\epsilon \left(i - \frac{\not P}{M_u + M_{\bar d}} \right) + \frac{2q\cdot\epsilon}{M_u + M_{\bar d}} \right\} F_\rho(q^2,P^2)\,, \end{aligned} \tag{8}$$

if we also neglect the small components F_2^π, F_2^ρ and F_7^ρ. Then the involved Bethe-Salpeter equations can be projected into the following simple form,

$$\begin{aligned} \left\{ q^2 + M_u M_{\bar d} \left[1 + \frac{P^2}{(M_u + M_{\bar d})^2} \right] \right\} F_\pi(q^2,P^2) &= \int \frac{d^4k}{(2\pi)^4} F_\pi(k^2,P^2) \\ &\quad \left\{ \frac{16}{3} \frac{g_s^2(q-k)}{(q-k)^2} + \frac{g_\pi^2(q-k)}{(q-k)^2 + m_\pi^2} \right\}\,, \\ \left\{ q^2 + M_u M_{\bar d} \left[1 + \frac{P^2}{(M_u + M_{\bar d})^2} \right] \right\} F_\rho(q^2,P^2) &= \int \frac{d^4k}{(2\pi)^4} F_\rho(k^2,P^2) \\ &\quad \left\{ \frac{8}{3} \frac{g_s^2(q-k)}{(q-k)^2} - \frac{g_\pi^2(q-k)}{(q-k)^2 + m_\pi^2} \right\}. \end{aligned} \tag{9}$$

If we take $q^2 = 0$ and $g_\pi^2 = 0$, and assume that there exists a physical solution, then

$$M_u M_{\bar d} \left[1 - \frac{M_{\pi/\rho}^2}{(M_u + M_{\bar d})^2} \right] F_{\pi/\rho}(0, -M_{\pi/\rho}^2) = \int \frac{d^4k}{(2\pi)^4} F_{\pi/\rho}(k^2, -M_{\pi/\rho}^2) G_{\pi/\rho}(0-k)\,, \tag{10}$$

where the $G_{\pi/\rho}(k)$ denotes the interacting kernels. In numerical calculations, we observe that the Bethe-Salpeter amplitude $F_{\pi/\rho}(k^2, -M_{\pi/\rho}^2)$ has the same sign in the region $k^2 \geq 0$,

$$1 - \frac{M_{\pi/\rho}^2}{(M_u + M_{\bar d})^2} = \int \frac{d^4k}{(2\pi)^4} \frac{F_{\pi/\rho}(k^2, -M_{\pi/\rho}^2)}{M_u M_{\bar d} F_{\pi/\rho}(0, -M_{\pi/\rho}^2)} G_{\pi/\rho}(0-k) > 0\,, \tag{11}$$

and obtain an simple relation (or constraint),

$$M_{\pi/\rho}^2 < (M_u + M_{\bar d})^2\,, \tag{12}$$

which survives for $q^2 > 0$ (although the relation is not explicit for $q^2 > 0$), i.e. the bound energy $E_{\pi/\rho}$ originates from the interacting kernel $G_{\pi/\rho}(k)$ and should be negative, $E_{\pi/\rho} = M_{\pi/\rho} - M_u - M_{\bar d} < 0$. The numerical calculations indicate that above arguments survive in the case $g_\pi^2 \neq 0$.

From Eq.(9), we can see explicitly that the one-gluon exchange force in the π channel is more attractive than that in the ρ channel due to the factors $\frac{16}{3}$ and $\frac{8}{3}$; furthermore, the one-pion exchange force is attractive in the π channel and repulsive in the ρ channel, the bound energy $|E_\pi| \gg |E_\rho|$ can be accounted for naturally.

We can introduce a parameter $\lambda(P^2)$ and solve above equations as an eigenvalue problem, the masses of the π and ρ mesons can be determined by the condition $\lambda(P^2 = -M^2_{\pi/\rho}) = 1$,

$$\begin{aligned}
\left\{q^2 + M_u M_{\bar{d}}\left[1+\frac{P^2}{(M_u+M_{\bar{d}})^2}\right]\right\} F_\pi(q^2,P^2) &= \lambda(P^2)\int \frac{d^4k}{(2\pi)^4} F_\pi(k^2,P^2) \\
&\quad \left\{\frac{16}{3}\frac{g_s^2(q-k)}{(q-k)^2} + \frac{g_\pi^2(q-k)}{(q-k)^2+m_\pi^2}\right\}, \\
\left\{q^2 + M_u M_{\bar{d}}\left[1+\frac{P^2}{(M_u+M_{\bar{d}})^2}\right]\right\} F_\rho(q^2,P^2) &= \lambda(P^2)\int \frac{d^4k}{(2\pi)^4} F_\rho(k^2,P^2) \\
&\quad \left\{\frac{8}{3}\frac{g_s^2(q-k)}{(q-k)^2} - \frac{g_\pi^2(q-k)}{(q-k)^2+m_\pi^2}\right\}.
\end{aligned} \tag{13}$$

3 Numerical results

The constituent quark masses of the u and $\bar{d}$ quarks are taken as $M_u = M_{\bar{d}} = 400\,\mathrm{MeV}$. The strength parameter A and the distribution width Λ are free parameters, we take the values $\Lambda = 200\,\mathrm{MeV}$ and $A = 146(105)$ for the one-gluon (one-pion) exchange. Other values of the Λ and A also work, we choose the present parameters for illustration.

We solve the Bethe-Salpeter equations as an eigenvalue problem numerically by direct iterations, and observe the convergent behaviors are very good. The numerical results for the Bethe-Salpeter amplitudes are shown in Fig.1. From the figure, we can see that the Bethe-Salpeter amplitudes center around zero momentum and extend to the energy scale about $q = 0.4\,\mathrm{GeV}$ and $0.7\,\mathrm{GeV}$ for the ρ and π mesons respectively, the stronger interactions in the π channel result in more stable bound state than that in the ρ channel, as the bound energies are $E_\pi = -660\,\mathrm{MeV}$ and $E_\rho = -25\,\mathrm{MeV}$, respectively. In numerical calculations, we observe that the one-pion exchange force plays an important role and should be taken into account.

With the following simple replacements in Eq.(13),

$$\begin{aligned}
M_{\bar{d}} &\rightarrow M_{\bar{s}} = 536\,\mathrm{MeV}, \\
g_\pi^2(k^2) &\rightarrow g_K^2(k^2) = A\left(\frac{k^2}{\widetilde{\Lambda}^2}\right)^2 \exp\left(-\frac{k^2}{\widetilde{\Lambda}^2}\right), \\
M_\pi &\rightarrow M_K = 494\,\mathrm{MeV},
\end{aligned} \tag{14}$$

where the $\widetilde{\Lambda} = \Lambda\left(\frac{M_K+M_{K^*}}{M_\pi+M_\rho}\right)^2$ denotes the $SU_V(3)$ breaking effect for the coupling constant, we can obtain the corresponding solutions for the pseudoscalar meson K and vector meson K^* with the eigenvalues $\lambda(P^2 = -M_K^2 = -(494\,\mathrm{MeV})^2) = 1$ and $\lambda(P^2 =$

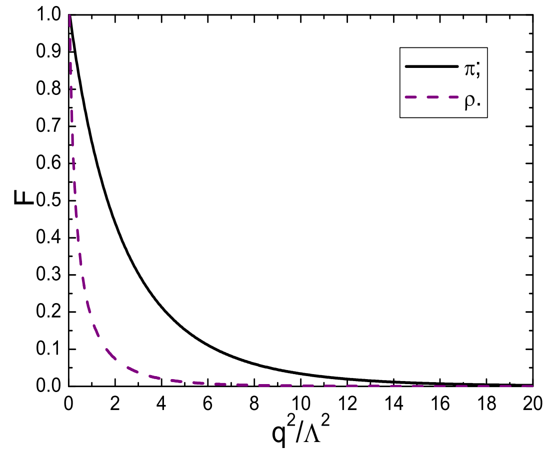

Figure 1: The Bethe-Salpeter amplitudes of the bound states.

$-M_{K^*}^2 = -(892\,\mathrm{MeV})^2) = 1$, respectively. The bound energies are $E_K = -442\,\mathrm{MeV}$ and $E_{K^*} = -44\,\mathrm{MeV}$, respectively, which indicate that there exists a more stable bound state in the pseudoscalar channel than that in the vector channel.

4 Conclusion

In this article, we take into account the one-pion exchange force besides the one-gluon exchange force to study the mass difference of the π and ρ mesons with the Bethe-Salpeter equation. After simplifying the involved Bethe-Salpeter equations, we observe that the one-gluon exchange force in the π channel is more attractive than that in the ρ channel, while the one-pion exchange force is attractive in the π channel and repulsive in the ρ channel, the bound energy $|E_\pi| \gg |E_\rho|$ can be accounted for naturally.

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